

On Hypotheses Testing for Poisson Processes. Singular Cases

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Abstract

We consider the problem of hypotheses testing in the situation where the first hypothesis is simple and the second one is one-sided local composite. We describe the choice of thresholds and the power functions of different tests when the intensity function of the inhomogeneous Poisson process has two different types of singularity: cusp and discontinuity. The asymptotic results are illustrated by the numerical simulations.

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1 Introduction

This is the second part of the study devoted to hypotheses testing problems in the case of observations of the inhomogeneous Poisson processes. The first part is devoted to the problems in the regular *smooth case* situation [3] and the second part is concerned . We suppose that the intensity function $\lambda(\vartheta, t)$ of the inhomogeneous Poisson process depends on the unknown parameter ϑ

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in non regular way. For example, the Fisher information is infinite. The basic hypothesis is always simple ($\vartheta = \vartheta_1$) and the alternative is local one-sided ($\vartheta > \vartheta_1$). In the first part it was described the asymptotic behavior of the score-function test (SCF), general likelihood ratio test (GLRT), Wald test (WT) and of two Bayesian tests (BT1, BT2). It was shown that the tests SCF, GLRT and WT are locally asymptotically uniformly most powerful. In the present work we study the asymptotic behavior of the GLRT, WT, BT1 and BT2 in two non regular (non smooth) situations. At particularly, we study the tests when the intensity functions has cusp-type singularity and jump-type singularity. In both cases the Fisher information is infinite. The local alternative is obtained by the following re-parametrization $\vartheta = \vartheta_1 + u\varphi_n, u > 0$. The rate of convergence $\varphi_n \rightarrow 0$ depends on the order of singularity. In the cusp case $\varphi_n \sim n^{-\frac{1}{2\kappa+1}}$ and in the discontinuous case $\varphi_n \sim n^{-1}$. Our goal is to describe the choice of the thresholds and the behavior of the power functions as $n \rightarrow \infty$. The important difference between smooth and non-smooth cases is due to the absence of the criteria of optimality. This leads to the situation when the comparison of the power functions can be done numerically only. That is why we present the results of numerical simulations of the limit power functions and the comparison of them with the power functions with small and large volumes of observations (small and large n).

Recall that $X = (X_t, t \geq 0)$; $X_0 = 0$ is an inhomogeneous Poisson process with intensity function $\lambda(t)$, if $X_0 = 0$, the increments of X on disjoint intervals are independent and distributed according to the Poisson law

$$\mathbf{P}\{X_t - X_s = k\} = \frac{\left(\int_s^t \lambda(t) dt\right)^k}{k!} \exp\left\{-\int_s^t \lambda(t) dt\right\}.$$

We suppose that the intensity function depends on some one-dimensional parameter, i.e., $\lambda(t) = \lambda(\vartheta, t)$ and the basic hypothesis is simple : $\vartheta = \vartheta_1$. The alternative is one-sided composite $\vartheta > \vartheta_1$.

The hypotheses testing problems for inhomogeneous Poisson processes were studied by many authors, see, for example, [12],[5], [3] and the references therein.

2 Preliminaries

We consider the model of n observations of independent inhomogeneous Poisson processes $X^n = (X_1, \dots, X_n)$, where $X_j = \{X_j(t), 0 \leq t \leq \tau\}$ and

$$\mathbf{E}_{\vartheta} X_j(t) = \Lambda(\vartheta, t) = \int_0^t \lambda(\vartheta, s) \, ds.$$

We use here the same notations as in [3]. Here ϑ is one-dimensional parameter and \mathbf{E}_{ϑ} is the mathematical expectation, when the true value is ϑ . The intensity function is supposed to be separated from zero on $[0, \tau]$, the measures corresponding to Poisson processes with the different values of ϑ are equivalent and the likelihood function is defined by the equality

$$L(\vartheta, X^n) = \exp \left\{ \sum_{j=1}^n \int_0^{\tau} \ln \lambda(\vartheta, t) \, dX_j(t) - n \int_0^{\tau} [\lambda(\vartheta, t) - 1] \, dt \right\}.$$

In non-regular situation we have no UMP test and it is interesting to compare the power functions of the different tests with the power function of the Neyman-Pearson test (N-PT). Let us recall the definition of N-PT. Suppose that we have two simple hypotheses $\mathcal{H}_1 : \vartheta = \vartheta_1$ and $\mathcal{H}_2 : \vartheta = \vartheta_2$ and our goal is to construct a test $\hat{\psi}_n(X^n)$ of size ε , i.e. a test with the fixed given probability of error of first kind: $\mathbf{E}_{\vartheta_1} \hat{\psi}_n(X^n) = \varepsilon$. As usual, the test $\hat{\psi}_n(X^n)$ is the probability to reject the hypothesis \mathcal{H}_1 and, of course, to accept the hypothesis \mathcal{H}_2 .

Let us denote the likelihood ratio statistic as

$$L(\vartheta_2, \vartheta_1, X^n) = L(\vartheta_2, X^n) / L(\vartheta_1, X^n).$$

Then by the Neyman-Pearson Lemma [13] the N-PT is

$$\hat{\psi}_n(X^n) = \begin{cases} 1, & \text{if } L(\vartheta_2, \vartheta_1, X^n) > b_{\varepsilon}, \\ q_{\varepsilon}, & \text{if } L(\vartheta_2, \vartheta_1, X^n) = b_{\varepsilon}, \\ 0, & \text{if } L(\vartheta_2, \vartheta_1, X^n) < b_{\varepsilon}. \end{cases}$$

The constants b_{ε} and q_{ε} are solutions of the equation

$$\mathbf{P}_{\vartheta_1}(L(\vartheta_2, \vartheta_1, X^n) > b_{\varepsilon}) + q_{\varepsilon} \mathbf{P}_{\vartheta_1}(L(\vartheta_2, \vartheta_1, X^n) = b_{\varepsilon}) = \varepsilon.$$

In this work we consider the construction of the tests in the following hypotheses testing problem

$$\begin{aligned} \mathcal{H}_1 & : & \vartheta &= \vartheta_1, \\ \mathcal{H}_2 & : & \vartheta &> \vartheta_1, \end{aligned} \tag{1}$$

i.e.; we have a simple hypothesis against one-sided composite alternative.

The log likelihood ratio function can be written as follows

$$\ln L(\vartheta, \vartheta_1, X^n) = \sum_{j=1}^n \int_0^\tau \ln \frac{\lambda(\vartheta, t)}{\lambda(\vartheta_1, t)} dX_j(t) - n \int_0^\tau [\lambda(\vartheta, t) - \lambda(\vartheta_1, t)] dt.$$

The power function of the test $\bar{\psi}_n(X^n)$ is $\beta(\vartheta, \bar{\psi}_n) = \mathbf{E}_{\vartheta} \bar{\psi}_n(X^n)$, $\vartheta > \vartheta_1$. Denote by \mathcal{K}_ε the class of tests $\bar{\psi}_n$ of asymptotic size ε :

$$\mathcal{K}_\varepsilon = \left\{ \bar{\psi}_n \quad : \quad \lim_{n \rightarrow \infty} \mathbf{E}_{\vartheta_1} \bar{\psi}_n(X^n) = \varepsilon \right\}.$$

In this work we study several tests which belong to the class \mathcal{K}_ε . To compare these tests by their power functions we consider, as usual, the approach of *close* or *contiguous* alternatives because for any fixed alternative the power functions of all tests converge to the same value 1. Let us put $\vartheta = \vartheta_1 + \varphi_n u$, where $\varphi_n = \varphi_n(\vartheta_1) > 0$. Here $\varphi_n \rightarrow 0$ and the rate of convergence depends on the type of the singularity of the intensity function.

Now the initial problem of hypotheses testing can be rewritten as follows

$$\begin{aligned} \mathcal{H}_1 & : & u &= 0, \\ \mathcal{H}_2 & : & u &> 0. \end{aligned} \tag{2}$$

The considered tests are usually of the form

$$\bar{\psi}_n = \mathbb{I}_{\{Y_n(X^n) > c_\varepsilon\}} + q_\varepsilon \mathbb{I}_{\{Y_n(X^n) = c_\varepsilon\}},$$

where the constant c_ε is defined with the help of the limit random variable Y (suppose that $Y_n \Rightarrow Y$ under hypothesis) by the following relation

$$\mathbf{E}_{\vartheta_1} \bar{\psi}_n = \mathbf{P}_{\vartheta_1} \{Y_n(X^n) > c_\varepsilon\} + q_\varepsilon \mathbf{P}_{\vartheta_1} \{Y_n(X^n) = c_\varepsilon\} \longrightarrow \mathbf{P}_{\vartheta_1} \{Y > c_\varepsilon\} = \varepsilon$$

if the limit random variable Y is continuous and by

$$\mathbf{P}_{\vartheta_1} \{Y > c_\varepsilon\} + q_\varepsilon \mathbf{P}_{\vartheta_1} \{Y = c_\varepsilon\} = \varepsilon$$

if Y has distribution function with jumps.

The corresponding power function we denoted as

$$\beta_n(\bar{\psi}_n, u) = \mathbf{E}_{\vartheta_1 + \varphi_n u} \bar{\psi}_n, \quad u > 0$$

and the coparison of the tests corresponds to the comparison of their limit power functions.

We consider two different models of close alternatives in non smooth cases. In both cases the function $\lambda(\vartheta, t)$ is not differentiable and the Fisher information is infinite. At particularly, we study the behavior of the tests in two situations. The first one is *cusp* case when the intensity function is continuous but not differentiable and the second is discontinuous intensity case. In both cases the intensity functions $\lambda(\vartheta, t)$ has no derivative at the point $t = \vartheta$.

Note that these statistical models were already studied before in the problems of parameter estimation (see [1] for the cusp type singularity and [9] for discontinuous type singularity) and here we will show the properties of the tests. The main tool, of course, is the limit behavior of the normalized likelihood ratio function, which was already studied before in the mentioned works but in a slightly different situations. The proofs given in this work are mainly based on the results presented in [1] and [9].

Recall that in the non regular cases considered in this work there is no LAUMP tests that is why the special attention is paid to the numerical simulations of the limit power functions.

3 Cusp type singularity

Suppose that the intensity function of the observed Poisson process is

$$\lambda(\vartheta, t) = a |t - \vartheta|^\kappa + h(t), \quad 0 \leq t \leq \tau, \quad \vartheta \in \Theta = [\vartheta_1, b],$$

where $\kappa \in (0, 1/2)$, $\vartheta_1 > 0$, $b < \tau$ and $h(\cdot)$ is a known positive bounded function. To study the local alternatives we introduce the normalizing function

$$\varphi_n = n^{-\frac{1}{2H}} \Gamma_{\vartheta_1}^{-\frac{1}{H}}, \quad \Gamma_{\vartheta_1}^2 = \frac{2a^2 B(\kappa + 1, \kappa + 1)}{h(\vartheta_1)} \left[\frac{1}{\cos(\pi\kappa)} - 1 \right],$$

where $B(\cdot, \cdot)$ is the *Beta-function* and $H = \kappa + \frac{1}{2}$ is the *Hurst parameter*.

The change of variables $\vartheta = \vartheta_1 + \varphi_n u$ reduces the initial problem (1) to the hypotheses testing problem (2).

Introduce the stochastic process

$$Z(u) = \exp \left\{ W^H(u) - \frac{|u|^{2H}}{2} \right\}, \quad u \in \mathcal{R}_+,$$

where $W^H(\cdot)$ is a *fractional Brownian motion*. Further let us define the random variable \hat{u} by the relation

$$Z(\hat{u}) = \sup_{u \geq 0} Z(u)$$

and introduce the reals h_ε and g_ε as solutions of the equations

$$\mathbf{P}(Z(\hat{u}) > h_\varepsilon) = \varepsilon, \quad \mathbf{P}(\hat{u} > g_\varepsilon) = \varepsilon. \quad (3)$$

Note that the likelihood ratio $Z(u)$ is the same as the likelihood ratio of the similar hypothesis problem ($u = 0$ against $u > 0$) in the case of observations $(Y(v), v \geq 0)$ of the following type

$$dY(v) = \mathbb{I}_{\{v < u\}} dv + dW^H(v), \quad v \geq 0.$$

The uniformly most powerfull (UMP) test in this problem does not exist and we have no asymptotically UMP tests in our problem.

3.1 GLRT

The GLRT is defined by the relations

$$\hat{\psi}_n(X^n) = \mathbb{I}_{\{Q(X^n) > h_\varepsilon\}},$$

where

$$Q(X^n) = \sup_{\vartheta > \vartheta_1} L(\vartheta, \vartheta_1, X^n) = L(\hat{\vartheta}_n, \vartheta_1, X^n)$$

and $\hat{\vartheta}_n$ is the maximum likelihood estimator.

Let us introduce the function

$$\hat{\beta}(u) = \mathbf{P} \left\{ \sup_{s > 0} \left[W^H(s) - \frac{|s - u|^{2H}}{2} \right] > \ln h_\varepsilon - \frac{|u|^{2H}}{2} \right\}, \quad u \geq 0.$$

The properties of this test are given in the following Proposition.

Proposition 1. *The GLRT $\hat{\psi}_n(X^n)$ belongs to \mathcal{K}_ε and its power function in the case of local alternatives $\vartheta = \vartheta_1 + \varphi_n u, u > 0$ has the following limit*

$$\beta(\hat{\psi}_n, u) \longrightarrow \hat{\beta}(u).$$

Proof. Introduce the normalized likelihood ratio process

$$Z_n(v) = L(\vartheta_1 + \varphi_n v, X^n) = \frac{L(\vartheta_1 + \varphi_n v, X^n)}{L(\vartheta_1, X^n)}, \quad v \in \mathbb{V}_n^+ = [0, \varphi_n^{-1}(b - \vartheta_1)],$$

and define the function $Z_n(v)$ lineary decreasing to zero on the interval $[\varphi_n^{-1}(b - \vartheta_1), \varphi_n^{-1}(b - \vartheta_1) + 1]$ and $Z_n(v) \equiv 0$ for all $v > \varphi_n^{-1}(b - \vartheta_1) + 1$. Now the random function $Z_n(v)$ is defined on \mathcal{R}_+ .

Let us fix some $d \leq 0$ and denote as $\mathcal{C}_d = \mathcal{C}_d(\mathcal{R}_d)$ the space of continuous functions on $\mathcal{R}_d = [d, \infty)$ with the property $\lim_{v \rightarrow \infty} z(v) = 0$. Introduce the uniform metric on this space and denote by \mathfrak{B} the corresponding borelian σ -algebra.

Let \mathbf{Q}_n and \mathbf{Q} be the measures induced on the measurable space $(\mathcal{C}_d, \mathfrak{B})$ by the stochastic processes $Z_n(v), v \geq d$ and $Z(v), v \geq d$. The continuity with probability 1 of the random functions $Z_n(v), v \geq d$ follows from the inequality (6) below and the Kolmogorov theorem.

When we study the likelihood ratio process under hypothesis \mathcal{H}_1 we take $d = 0$ and consider the corresponding measurable space $(\mathcal{C}_0, \mathfrak{B})$. Under alternative \mathcal{H}_2 with $\vartheta = \vartheta_u = \vartheta_1 + \varphi_n u$ we will use this space with $d = -u$.

Suppose that we already proved the following weak convergence in $(\mathcal{C}_0, \mathfrak{B})$ (under hypothesis \mathcal{H}_1)

$$\mathbf{Q}_n \Longrightarrow \mathbf{Q}. \quad (4)$$

Then the distributions of all continuous in the uniform metric functionals $\Phi(Z_n)$ converge to the distribution of $\Phi(Z)$. At particularly, if we take

$$\Phi(z) = \sup_{v \geq 0} z(v) - h_\varepsilon,$$

then this weak convergence gives us the following relations

$$\begin{aligned} \mathbf{P}_{\vartheta_1}^{(n)} \left\{ \sup_{\vartheta > \vartheta_1} L(\vartheta, \vartheta_1, X^n) > h_\varepsilon \right\} &= \mathbf{P}_{\vartheta_1}^{(n)} \left\{ \sup_{v > 0} Z_n(v) > h_\varepsilon \right\} \\ &\longrightarrow \mathbf{P}_{\vartheta_1} \left\{ \sup_{v > 0} Z(v) > h_\varepsilon \right\} = \mathbf{P} \{ Z(\hat{u}) > h_\varepsilon \} = \varepsilon. \end{aligned}$$

Therefore the test $\hat{\psi}_n \in \mathcal{K}_\varepsilon$.

We do not know an analytical solution of this equation that is why we turn to the simulation method and choose the constant h_ε from numerical simulations. Note that $h_\varepsilon = h_\varepsilon(H)$ and does not depend on Γ_{ϑ_1} .

To study the power function we consider the same likelihood ratio process but under alternative $\vartheta_u = \vartheta_1 + \varphi_n u$. We can write

$$\begin{aligned} Z_n(v) &= \frac{L(\vartheta_1 + \varphi_n v, X^n)}{L(\vartheta_1, X^n)} = \frac{L(\vartheta_u, X^n)}{L(\vartheta_1, X^n)} \frac{L(\vartheta_1 + \varphi_n v, X^n)}{L(\vartheta_u, X^n)} \\ &= \left(\frac{L(\vartheta_u - \varphi_n u, X^n)}{L(\vartheta_u, X^n)} \right)^{-1} \frac{L(\vartheta_u + (v - u)\varphi_n, X^n)}{L(\vartheta_u, X^n)} \\ &= \tilde{Z}_n(-u)^{-1} \tilde{Z}_n(v - u), \quad v \geq 0 \end{aligned}$$

with obvious notation. The difference between $Z_n(\cdot)$ and $\tilde{Z}_n(\cdot)$ is that the *true value* in the first case is fixed ϑ_1 and in the second case it *runs* $\vartheta_u = \vartheta_1 + \varphi_n u$. The random variables $\tilde{Z}_n(-u)$ converge in distribution to $Z(-u)$. For the random process $\tilde{Z}_n(v-u), v \geq 0$ we have a similar joint convergence: for any fixed $v \geq 0$

$$\left(\tilde{Z}_n(-u), \tilde{Z}_n(v-u)\right) \Longrightarrow (Z(-u), Z(v-u)).$$

Let us denote by $\tilde{\mathbf{Q}}_n$ and $\tilde{\mathbf{Q}}$ the measures induced by the processes $(\tilde{Z}_n(v), v \geq -u)$ and $(\tilde{Z}(v), v \geq -u)$ in the measurable space $(\mathcal{C}_{-u}, \mathfrak{B})$ and suppose that we already proved the weak convergence

$$\tilde{\mathbf{Q}}_n \Longrightarrow \tilde{\mathbf{Q}}. \quad (5)$$

Then for the power function we can write

$$\begin{aligned} & \mathbf{P}_{\vartheta_u}^{(n)} \left\{ \sup_{v>0} Z_n(v) > h_\varepsilon \right\} \\ &= \mathbf{P}_{\vartheta_u}^{(n)} \left\{ \tilde{Z}_n(-u)^{-1} \sup_{v>0} \frac{L(\vartheta_u + (v-u)\varphi_n, X^n)}{L(\vartheta_u, X^n)} > h_\varepsilon \right\} \\ &\longrightarrow \mathbf{P}_{\vartheta_1} \left\{ (Z(-u))^{-1} \sup_{v>0} \exp \left\{ W^H(v-u) - \frac{|v-u|^{2H}}{2} \right\} > h_\varepsilon \right\} \\ &= \mathbf{P} \left\{ \sup_{s>0} \left[-W^H(-u) + W^H(s-u) - \frac{|s-u|^{2H}}{2} + \frac{|u|^{2H}}{2} \right] > \ln h_\varepsilon \right\} \\ &= \mathbf{P} \left\{ \sup_{s>0} \left[W^H(s) - \frac{|s-u|^{2H}}{2} \right] > \ln h_\varepsilon - \frac{|u|^{2H}}{2} \right\} = \hat{\beta}(u). \end{aligned}$$

This power function is obtained below with the help of numerical simulations (see section 3.4).

To finish the proof we need to verify the convergence (5). To do this we follow the proof of the convergence (4) given in [1]. Moreover, we present here the uniform w.r.t. ϑ_1 version of this convergence, i.e., we suppose that $\vartheta_1 = \vartheta(n)$, where $\vartheta(n) \in \mathbb{K}$ and \mathbb{K} is an arbitrary compact in Θ .

Introduce following relations.

1. *The finite-dimensional distributions of the random process $\tilde{Z}_n(v), v \geq -u$ converge to the finite-dimensional distributions of $Z(v), v \geq -u$ uniformly in $\vartheta \in \mathbb{K}$.*

2. *There exists a positive constant C such that*

$$\sup_{\vartheta_1 \in \mathbb{K}} \mathbf{E}_{\vartheta_u} \left| \tilde{Z}_n^{1/2}(v_2) - \tilde{Z}_n^{1/2}(v_1) \right|^2 \leq C |v_2 - v_1|^{2H}, \quad v_1, v_2 \geq -u. \quad (6)$$

3. *There exists a positive constant c such that*

$$\sup_{\vartheta_1 \in \mathbb{K}} \mathbf{E}_{\vartheta_u} \tilde{Z}_n^{1/2}(v) \leq \exp \left\{ -c |v - u|^{2H} \right\}. \quad (7)$$

The proofs of these relations are slight modifications of the proofs given in [1]. Note that the characteristic function of the vector

$$\tilde{Z}_n(u), \tilde{Z}_n(v_1), \dots, \tilde{Z}_n(v_k)$$

can be written explicitly and the convergence of the characteristic functions to the characteristic function of the limit process can be done directly (see [1], Lemma 5).

The inequalities (6) and (7) follow from the [1], Lemma 6 and Lemma 7 respectively. These relations allow us to apply the Theorem 1.10.1 in [7], where the weak convergence (5) under conditions 1-3 was proved. Note that the convergence (4) is a particular case of (5) which corresponds to $u = 0$.

3.2 Wald test

The MLE $\hat{\vartheta}_n$ is defined by the equation

$$L(\hat{\vartheta}_n, \vartheta_1, X^n) = \sup_{\vartheta \in \Theta} L(\vartheta, \vartheta_1, X^n).$$

The test of Wald (WT) has the following form

$$\psi_n^o(X^n) = \mathbb{I}_{\{\varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_1) > g_\varepsilon\}},$$

where the threshold g_ε is solution of the equation (3). Introduce as well the random variable \hat{u}_* as solution of the equation

$$\sup_{v \geq -u} Z(v) = Z(\hat{u}_*).$$

Proposition 2. *The WT belongs to \mathcal{K}_ε and its power function in the case of local alternatives $\vartheta = \vartheta_1 + \varphi_n u, u > 0$ has the following limit*

$$\beta(\psi_n^o, u) \longrightarrow \beta^o(u) = \mathbf{P}(\hat{u}_* > g_\varepsilon - u).$$

Proof. The MLE (under hypothesis \mathcal{H}_1) converges in distribution

$$\varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_1) \Longrightarrow \hat{u}.$$

Hence $\psi_n^o \in \mathcal{K}_\varepsilon$. For the proof see [1]. Recall that this convergence is a consequence of the weak convergence (4). Moreover, it is uniform w.r.t. $\vartheta_1 \in \mathbb{K}$. Let us study this estimator under alternative $\vartheta_u = \vartheta_1 + \varphi_n u$. We have

$$\begin{aligned} \mathbf{P}_{\vartheta_u} \left(\varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_u) < x \right) &= \mathbf{P}_{\vartheta_u} \left(\sup_{\varphi_n^{-1}(\theta - \vartheta_u) < x} L(\theta, \vartheta_u, X^n) > \sup_{\varphi_n^{-1}(\theta - \vartheta_u) \geq x} L(\theta, \vartheta_u, X^n) \right) \\ &= \mathbf{P}_{\vartheta_u} \left(\sup_{-u \leq v < x} \tilde{Z}_n(v) > \sup_{v \geq x} \tilde{Z}_n(v) \right) \longrightarrow \mathbf{P} \left(\sup_{-u \leq v < x} Z(v) > \sup_{v \geq x} Z(v) \right) \\ &= \mathbf{P}(\hat{u}_* < x). \end{aligned}$$

Here

$$\tilde{Z}_n(v) = \frac{L(\vartheta_u + \varphi_n v, X^n)}{L(\vartheta_u, X^n)}, \quad v \geq -u.$$

The limit of the power function of this test for the local alternative $\vartheta_u = \vartheta_1 + u\varphi_n$ we obtain from this convergence as follows:

$$\begin{aligned} \beta(\psi_n^o, u) &= \mathbf{P}_{\vartheta_u} \left\{ \varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_u) + u > g_\varepsilon \right\} \\ &\longrightarrow \mathbf{P} \{ \hat{u}_* > g_\varepsilon - u \} = \beta^o(u). \end{aligned}$$

The threshold g_ε and the power function $\beta^o(u)$ are obtained by the numerical simulations below.

3.3 Bayesian tests

Suppose that the parameter ϑ is a random variable with the density *a priori* $p(\theta)$, $\vartheta_1 \leq \theta < b$. This function is supposed to be continuous and positive.

We consider two tests. The first bayesian test is like WT but is based on the bayesian estimator and the second test is based on the the averaged likelihood ratio.

First test. Suppose that the loss function is quadratic, then the bayesian estimator $\tilde{\vartheta}_n$ is the following conditional expectation

$$\tilde{\vartheta}_n = \int_{\vartheta_1}^b \theta p(\theta | X^n) d\theta = \frac{\int_{\vartheta_1}^b \theta p(\theta) L(\theta, X^n) d\theta}{\int_{\vartheta_1}^b p(\theta) L(\theta, X^n) d\theta}.$$

Introduce the test (BT1)

$$\tilde{\psi}_n(X^n) = \mathbb{I}_{\{\varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1) > k_\varepsilon\}},$$

where the constant k_ε is solution of the equation

$$\mathbf{P}(\tilde{u} > k_\varepsilon) = \varepsilon, \quad \tilde{u} = \frac{\int_0^\infty v Z(v) dv}{\int_0^\infty Z(v) dv}.$$

Introduce as well the function

$$\tilde{\beta}(u) = \mathbf{P}(\tilde{u}_* > k_\varepsilon), \quad \tilde{u}_* = \frac{\int_{-u}^\infty v Z(v) dv}{\int_{-u}^\infty Z(v) dv}, \quad u \geq 0.$$

Proposition 3. *The BT1 $\tilde{\psi}_n(X^n) \in \mathcal{K}_\varepsilon$ and its power function*

$$\beta_n(\tilde{\psi}_n, u) \longrightarrow \tilde{\beta}(u)$$

Proof. The bayesian estimator $\tilde{\vartheta}_n$ is consistent and has the following limit distribution (under hypothesis \mathcal{H}_1)

$$\varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1) \Longrightarrow \tilde{u}.$$

For the proof see [1]. Hence $\tilde{\psi}_n(X^n) \in \mathcal{K}_\varepsilon$.

For the power function we have

$$\beta(\tilde{\psi}_n, u) = \mathbf{P}_{\vartheta_u} \left\{ \varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1) > k_\varepsilon \right\} = \mathbf{P}_{\vartheta_u} \left\{ \varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_u) > k_\varepsilon - u \right\}.$$

Let us study the normalized difference $\tilde{u}_n = \varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_u)$. We can write (below $\theta = \vartheta_u + \varphi_n v$)

$$\begin{aligned} & \int_{\vartheta_1}^b \theta p(\theta) L(\theta, \vartheta_u, X^n) d\theta \\ &= \varphi_n \int_{-u}^{\varphi_n^{-1}(b - \vartheta_u)} (\vartheta_u + \varphi_n v) p(\vartheta_u + \varphi_n v) L(\vartheta_u + \varphi_n v, \vartheta_u, X^n) dv \\ &= \varphi_n \int_{-u}^{\varphi_n^{-1}(b - \vartheta_u)} (\vartheta_u + \varphi_n v) p(\vartheta_u + \varphi_n v) \tilde{Z}_n(v) dv. \end{aligned}$$

Hence

$$\tilde{u}_n = \frac{\int_{-u}^{\varphi_n^{-1}(b - \vartheta_u)} v p(\vartheta_u + \varphi_n v) \tilde{Z}_n(v) dv}{\int_{-u}^{\varphi_n^{-1}(b - \vartheta_u)} p(\vartheta_u + \varphi_n v) \tilde{Z}_n(v) dv} \Longrightarrow \frac{\int_{-u}^\infty v Z(v) dv}{\int_{-u}^\infty Z(v) dv} = \tilde{u}_*,$$

because $p(\vartheta_u + \varphi_n v) \rightarrow p(\vartheta_1) > 0$ and $\tilde{Z}_n(v) \implies Z(v)$. The detailed proof is based on the properties 1-3 of the likelihood ratio (see [1] or [7], Theorem 1.10.2).

Second test. Let us introduce the BT2

$$\psi_n^*(X^n) = \mathbb{I}_{\{R_n(X^n) > m_\varepsilon\}}, \quad R_n(X^n) = \frac{\tilde{L}_n(X^n)}{p(\vartheta_1)\varphi_n}.$$

Here

$$\tilde{L}(X^n) = \int_{\vartheta_1}^b L(\theta, \vartheta_1, X^n) p(\theta) d\theta$$

and m_ε is solution of the equation

$$\mathbf{P} \left\{ \int_0^\infty \exp \left\{ W^H(v) - \frac{v^{2H}}{2} \right\} dv > m_\varepsilon \right\} = \varepsilon.$$

Define the function

$$\beta^*(u) = \mathbf{P} \left(Z(-u)^{-1} \int_{-u}^\infty Z(v) dv > m_\varepsilon \right)$$

Proposition 4. *The BT2 $\psi_n^*(X^n) \in \mathcal{K}_\varepsilon$ and its power function under the local alternatives $\vartheta_u = \vartheta_1 + \varphi_n u$ converges to the following limit*

$$\beta_n(\psi_n^*, u) \longrightarrow \beta^*(u).$$

Proof. Let us recall how this test was obtained. Introduce the mean error $\bar{\alpha}(\bar{\psi}_n)$ under alternative \mathcal{H}_2 of an arbitrary test $\bar{\psi}_n$

$$\bar{\alpha}(\bar{\psi}_n) = \int_{\vartheta_1}^b \mathbf{E}_\theta \bar{\psi}_n(X^n) p(\theta) d\theta = \mathbb{E} \bar{\psi}_n,$$

where \mathbb{E} is the double mathematical expectation, i.e., the expectation with respect to the measure

$$\mathbb{P}(X^n \in A) = \int_{\vartheta_1}^b \mathbf{P}_\theta(X^n \in A) p(\theta) d\theta.$$

If we consider the problem of the minimization of this mean error we reduce the initial hypotheses testing problem to the problem of testing two simple hypotheses

$$\begin{aligned} \mathcal{H}_1 &: X^n \sim \mathbf{P}_{\vartheta_1}, \\ \mathcal{H}_2 &: X^n \sim \mathbb{P}. \end{aligned}$$

Then by the Neyman-Pearson Lemma the most powerfull test in the classe \mathcal{K}_ε (which minimizes the mean error $\bar{\alpha}(\bar{\psi}_n)$) is

$$\bar{\psi}_n^*(X^n) = \mathbb{I}_{\{\tilde{L}_n(X^n) > r_\varepsilon\}}, \quad \tilde{L}_n(X^n) = \frac{d\mathbb{P}}{d\mathbf{P}_{\vartheta_1}}(X^n),$$

where the average likelihood ratio

$$\tilde{L}(X^n) = \varphi_n \int_0^{\varphi_n^{-1}(\beta - \vartheta_1)} Z_n(v) p(\vartheta_1 + u\varphi_n) dv$$

and r_ε is choosen from the condition $\bar{\psi}_n^* \in \mathcal{K}_\varepsilon$. Therefore the BT2 $\psi_n^*(X^n)$ coincides with $\bar{\psi}_n^*(X^n)$ if we put $r_\varepsilon = m_\varepsilon p(\vartheta_1) \varphi_n$.

In the proof of the convergence in distribution of the bayesian estimator $\tilde{u}_n = \varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1)$ it is shown (see Theorem 1.10.2 in [7] and [1]) that

$$\begin{aligned} \varphi_n^{-1} \tilde{L}(X^n) &\Rightarrow p(\vartheta_1) \int_0^\infty \exp\left\{W^H(v) - \frac{v^{2H}}{2}\right\} dv \\ &= p(\vartheta_1) \int_0^\infty \exp\left\{W^H(v) - \frac{v^{2H}}{2}\right\} dv. \end{aligned}$$

Therefore

$$R_n(X^n) \Rightarrow \int_0^\infty Z(v) dv$$

and the test $\psi_n^*(X^n)$ belongs to the class $\in \mathcal{K}_\varepsilon$.

Using the similar arguments we can verify the convergence

$$R_n(X^n) \Rightarrow Z(-u)^{-1} \int_{-u}^\infty Z(v) dv$$

under alternative ϑ_u .

3.4 Simulations

Let us consider the following example. We observe n independent realizations of inhomogeneous Poisson process $X^n = (X_1, \dots, X_n)$, where $X_j = \{X_j(t), t \in [0, 2]\}$, $j = 1, \dots, n$. The intensity function of this processes is

$$\lambda(\vartheta, t) = 2 - |t - \vartheta|^{0.4}, \quad 0 \leq t \leq 2,$$

where the parameter $\vartheta \in [\frac{1}{2}, 2)$. We take $\vartheta_1 = 1.5$ as the value of the basic hypothesis \mathcal{H}_1 . Of course it is sufficient to have simulations for the values

$\vartheta \in [1.5, 2]$ but we consider the wider interval to show the behavior of the likelihood ratio on the both sides of true value. The Hurst parameter is $H = 0.9$ and the constant $\Gamma_{\vartheta_1}^2 = B(1.4, 1.4) \left[\frac{1}{\cos(0.4\pi)} - 1 \right] \approx 1.027$.

The realization of the normalized likelihood ratio $Z_n(v), v \in [-5, 5]$ and its zoom $Z_n(v), v \in [0.1, 0.5]$ under the basic hypothesis are given on the Fig. 1.

Here Fig. 1

To find the thresholds of the GLRT and WT we need to calculate the point of maximum and the maximal value of this function. In the case of the chosen intensity function the maximum is attained at one of the cusp points of the observations.

It is interesting to note that if the intensity function has the same singularity but with a different sign $\lambda(\vartheta, t) = 0.5 + |t - \vartheta|^{0.4}$, then to find the maximum is much more difficult (see Fig. 2).

Here Fig. 2

The threshold of the GLRT is obtained by simulating $M = 10^5$ r.v.s of $Z^i(v), v \in [0, 20], i = 1, \dots, M$ (when $v > 20$ the value of $Z(v)$ is negligible), calculating for each of them the quantity $\sup_v Z^i(v)$ and taking $(1 - \varepsilon)M$ -th greatest between them.

The thresholds of the mentioned tests are presented on the following table.

ε	0.01	0.05	0.10	0.2	0.4	0.5
$\ln h_\varepsilon$	2.959	1.641	1.081	0.559	0.159	0.068
g_ε	3.041	1.996	1.521	0.950	0.333	0.166
k_ε	2.864	2.0776	1.720	1.365	1.005	0.885

Table 1: Thresholds of GLRT, WT and BT1.

For evaluation of the power function we calculate the frequency of accepting the alternative hypothesis ϑ_u

$$\beta_n(u) \approx \frac{1}{N} \sum_{i=1}^N \mathbb{I}_{\left\{ \sup_{v>0} Z_{n,i}(v) > h_\varepsilon \right\}}.$$

We can see (Fig 3) that, like the regular case, for the small values of u the power function of WT converge more slowly than that of GLRT, but still quicker than that of BT1. When u is large, the power function of BT1

converge more quickly than WT, and the power function of GLRT converge the most slowly.

Here Fig. 3

Our goal is to compare the limit power functions of the three studied tests with the help of numerical simulations because the analytic expressions for these power functions are not yet available. It will be interesting to see as well the limit power function of the Neyman-Pearson Test (N-PT) constructed in the problem of the testing of two simple hypotheses as follows. Let us fix an alternative $\vartheta_2 = \vartheta_1 + u_*\varphi_n > \vartheta_1$ and consider the hypotheses testing problem

$$\begin{aligned}\mathcal{H}_1 &: u = 0, \\ \mathcal{H}_2 &: u = u_*.\end{aligned}$$

The Neyman-Pearson test is

$$\psi_n^*(X^n) = \mathbb{I}_{\{Z_n(u_*) > d_\varepsilon\}},$$

where the threshold d_ε is la solution of the equation

$$\mathbf{P}_{\vartheta_1}(Z(u_*) > d_\varepsilon) = \varepsilon.$$

Recall that $Z_n(u_*) \Rightarrow Z(u_*)$ and

$$Z(u_*) = \exp \left\{ W^H(u_*) - \frac{u_*^{2H}}{2} \right\}.$$

Hence

$$\mathbf{P}_{\vartheta_*}(Z(u_*) > d_\varepsilon) = \mathbf{P} \left\{ W^H(u_*) - \frac{u_*^{2H}}{2} > \ln d_\varepsilon \right\} = \mathbf{P} \left(\zeta > \frac{\ln d_\varepsilon + \frac{u_*^{2H}}{2}}{u_*^H} \right),$$

and

$$d_\varepsilon = e^{z_\varepsilon u_*^H - \frac{u_*^{2H}}{2}}, \quad \mathbf{P}(\zeta > z_\varepsilon) = \varepsilon, \quad \zeta \sim \mathcal{N}(0, 1).$$

Of course, it is impossible indeed to have N-PT because the value of u_* under alternative is unknown, but as this test is the most powerful in the class \mathcal{K}_ε its power function shows an upper bound for powers of all tests. The distance between it and the power functions of studied tests provides useful information.

To study the likelihood ratio function under alternative we write

$$Z_n(u_*) = \frac{L(\vartheta_1 + u_*\varphi_n, X^n)}{L(\vartheta_1, X^n)} = \left(\frac{L(\vartheta_1 + u_*\varphi_n - u_*\varphi_n, X^n)}{L(\vartheta_1 + u_*\varphi_n, X^n)} \right)^{-1}.$$

For the power function of N-PT we obtain

$$\begin{aligned}\beta_n(u_*) &= \beta(\psi_n^*, u_*) = \mathbf{P}_{\vartheta_1 + u_* \varphi_n} (Z_n(u_*) > d_\varepsilon) \longrightarrow \beta(u_*) \\ &= \mathbf{P}_{\vartheta_*} ((Z(-u_*))^{-1} > d_\varepsilon) = \mathbf{P}_{\vartheta_1} \left(\exp \left\{ -W^H(-u_*) + \frac{u_*^{2H}}{2} \right\} > d_\varepsilon \right).\end{aligned}$$

and hence

$$\beta(u_*) = \mathbf{P} \left(\zeta > \frac{\ln d_\varepsilon - \frac{u_*^{2H}}{2}}{u_*^H} \right) = \mathbf{P} (\zeta > z_\varepsilon - u_*^H).$$

Here Fig. 4

We can see that, the power function of GLRT is the closest one to the power function of NP-T. When u is small, the power function of WT is lower than BT1. It becomes closer with that of GLRT when u increases. At the same time, the power function of BT1 will become the lowest one. We also mention that for the power function of BT1 arrives faster to 1 than the others (see Fig. 4).

4 Discontinuous intensity

Let us consider the similar hypotheses testing problem but in the case of inhomogeneous Poisson process with discontinuous intensity function. Suppose that we have n independent observations $X^n = (X_1, \dots, X_n)$ of the inhomogeneous Poisson processes $X_j = (X_j(t), 0 \leq t \leq \tau)$ with the intensity function $\lambda(\vartheta, t), 0 \leq t \leq \tau$ and this intensity function satisfies the following condition.

***S.** The intensity function $\lambda(\vartheta, t) = \lambda(t - \vartheta)$, where the parameter $\vartheta \in \Theta = (\vartheta_1, b) \subset (0, \tau)$, the function $\lambda(s), s \in [-b, \tau - \vartheta_1]$ is continuously differentiable everywhere except at the point $t_* \in (0, \tau)$ and this function has a jump $r \neq 0$ at the point t_* .*

Therefore the intensity function $\lambda(\vartheta, t)$ has jump at the instant $t = t_* + \vartheta$. We have to test the hypotheses

$$\begin{aligned}\mathcal{H}_1 &: \quad \vartheta = \vartheta_1, \\ \mathcal{H}_2 &: \quad \vartheta > \vartheta_1.\end{aligned}$$

We are interested by the same tests as before (GLRT, WT, BT) and our goal is to chose the thresholds such that these tests belong to the class \mathcal{K}_ε .

Let us denote $\lambda(t_*+) = \lambda_+$, $\lambda(t_*-) = \lambda_-$ and $\rho = \frac{\lambda_-}{\lambda_+}$. To compare their power functions we turn to the close alternatives and in this problem we take $\vartheta = \vartheta_1 + u\varphi_n$; $\varphi_n = \frac{1}{n\lambda_+}$. The initial problem is reduced to the following one

$$\begin{aligned}\mathcal{H}_1 &: u = 0, \\ \mathcal{H}_2 &: u > 0.\end{aligned}$$

Recall that the normalized likelihood ratio

$$L(\vartheta_1 + \varphi_n v, \vartheta_1, X^n) = \frac{L(\vartheta_1 + \varphi_n v, X^n)}{L(\vartheta_1, X^n)}, \quad v \in [0, n\lambda_+(\tau - t_* - \vartheta_1)]$$

under hypothesis \mathcal{H}_1 converges to the process

$$Z(v) = \exp\{\ln \rho x_*(v) - (\rho - 1)v\}, \quad v \geq 0,$$

where $x_*(v), v \geq 0$ is the Poisson process of unit intensity [8].

The limit likelihood ratio under alternative \mathcal{H}_2 is

$$Z(v, u) = \exp\{\ln \rho x_*(v \wedge u) - (\rho - 1)(v \wedge u)\}, \quad v \geq 0,$$

(see below) i.e., it is the same as the likelihood ratio in the problem of hypotheses testing by observations of Poisson process $x_*(v), v \geq 0$ with the switching intensity function

$$\rho \mathbb{I}_{\{v < u\}} + \mathbb{I}_{\{v \geq u\}}, \quad v \geq 0. \quad (8)$$

To compare the power functions of different tests, we consider this likelihood ratio under (close) alternative $u > 0$.

4.1 Weak convergence

The GLRT, WT, BT are some functionals of the likelihood $L(\vartheta, X^n)$. As it was shown above all these tests can be written as functionals of the normalized likelihood ratio $Z_n(\cdot)$. Therefore as in Regular and cusp cases we have to prove the weak convergence of the measures induced by the normalized likelihood ratio under hypothesis (to find the thresholds) and under alternative (to describe the power functions).

Let \mathbb{D}_0 be the space of functions $z(\cdot)$ on $\mathcal{R}_+ = [0, +\infty)$ which do not have discontinuities of the second kind and which are such that $\lim_{v \rightarrow \infty} z(v) = 0$. We suppose that the functions $z(\cdot)$ are cadlag; that is, the left limit $z(t-) =$

$\lim_{s \nearrow t} z(s)$ exists and the right limit $z(t-) = \lim_{s \searrow t} z(s)$ exists and equals to $z(t)$.

Introduce the distance between two function $z_1(\cdot)$ and $z_2(\cdot)$ as follows

$$d(z_1, z_2) = \inf_{\nu} \left[\sup_{u \in \mathcal{R}_+} |z_1(u) - z_2(\nu(u))| + \sup_{u \in \mathcal{R}_+} |u - \nu(u)| \right],$$

where \inf is taken over all monotone, continuous, one-to-one mappings $\nu(\cdot) : \mathcal{R}_+ \rightarrow \mathcal{R}_+$. Let us denote

$$\Delta_h(z) = \sup_{u \in \mathcal{R}_+} \sup_{u \in \delta} \left\{ \min \left[|z(u') - z(u)|, |z(u'') - z(u)| \right] \right\} + \sup_{|u| > 1/h} |z(u)|,$$

where the interval $\delta = [u', u''] \subseteq [u - h, u + h]$.

Suppose that we have a sequence of stochastic processes $(Y_n)_{n \geq 1}$, $Y_n = \{Y_n(u), u \in [0, +\infty)\}$ and a process $Y_0 = \{Y_0(u), u \in [0, +\infty)\}$ such that the realizations of these processes belong to the space \mathbb{D}_0 and denote by $\mathbf{Q}_{\vartheta}^{(n)}$ and \mathbf{Q}_{ϑ} the distributions induced on the measurable space $(\mathbb{D}_0, \mathcal{B})$ by these processes respectively, i.e., we suppose that these distributions depend on the parameter $\vartheta \in \Theta$. Here \mathcal{B} is the borelian σ -algebra of the metric space \mathbb{D}_0 .

Theorem 1. *Let the finite dimensional distributions of the process Y_n converge to the finite dimensional distributions of the process Y_0 as $n \rightarrow \infty$ uniformly for $\vartheta \in \Theta$ and for any $\delta > 0$*

$$\lim_{h \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{\vartheta \in \Theta} \mathbf{Q}_{\vartheta}^n \{ \Delta_h(Y_n) > \delta \} = 0. \quad (9)$$

Then $\mathbf{Q}_{\vartheta}^{(n)} \rightarrow \mathbf{Q}_{\vartheta}$ uniformly in $\vartheta \in \Theta$ as $n \rightarrow \infty$.

For the proof see [6], Theorem 9.5.2.

Recall that such weak convergence of the likelihood ratio process $Z_n(\cdot)$ for the discussed model of inhomogeneous Poisson process was already established in [8], Section 4.4 (see as well [9], Chapter 5 for similar results). The given there proof corresponds to the weak convergence in the space $(\mathbb{D}_0, \mathcal{B})$ of $Z_n(\cdot)$ under hypothesis. Under alternative the limit process is different and we give here the related estimates which allow to describe the power functions.

Let us denote the measures induced by the realizations of $Z_n(v), v \geq 0$ and $Z(v, u), v \geq 0$ in the measurable space $(\mathbb{D}_0, \mathcal{B})$ as $\mathbf{Q}_{\vartheta}^{(n)}$ and \mathbf{Q}_{ϑ} respectively.

Proposition 5. *Let the condition **S** be fulfilled. Then we have the convergence*

$$\mathbf{Q}_{\vartheta}^{(n)} \rightarrow \mathbf{Q}_{\vartheta}. \quad (10)$$

The proof is based on several lemmata, where we verify the convergence of finite-dimensional distributions and (9). As in [8] we follow the main steps of the similar convergence proved by Ibragimov and Khasminskii [7] in the case of i.i.d. observations. At particularly, we show that in our case the likelihood ratio process has the same properties.

Lemma 1. *Let the condition **S** be fulfilled. Then, under alternative \mathcal{H}_2 , the finite-dimensional distributions of the process $Z_n(v), v \geq 0$ converge to those of the process $Z(v, u), v \geq 0$.*

Proof. The characteristic function of $\ln Z_n(v)$ can be written as follows (see [8]):

$$\begin{aligned} & \mathbf{E}_{\vartheta_1 + u\varphi_n} \exp \{i\mu \ln Z_n(v)\} \\ &= \exp \left[n \int_0^\tau \left(\exp \left(i\mu \ln \frac{\lambda(t - \vartheta_1 - v\varphi_n)}{\lambda(t - \vartheta_1)} \right) - 1 \right) \lambda(t - \vartheta_1 - u\varphi_n) dt - \right. \\ & \quad \left. - i n \mu \int_0^\tau (\lambda(t - \vartheta_1 - v\varphi_n) - \lambda(t - \vartheta_1)) dt \right] \\ &= \exp \left(n \int_0^\tau A_n(v, t) dt - i n \mu \int_0^\tau B_n(v, u, t) dt \right) \end{aligned}$$

where we denoted

$$\begin{aligned} A_n(v, t) &= \left[\exp \left(i\mu \ln \frac{\lambda_v}{\lambda_0} \right) - 1 - i\mu \ln \frac{\lambda_v}{\lambda_0} \right] \lambda_v, \\ B_n(v, u, t) &= \lambda_v - \lambda_0 - \lambda_0 \ln \frac{\lambda_v}{\lambda_0} + (\lambda_0 - \lambda_u) \ln \frac{\lambda_v}{\lambda_0} \end{aligned}$$

and $\lambda_v = \lambda(t - \vartheta_1 - v\varphi_n)$ with corresponding λ_0 and λ_u .

We consider two cases, $v \leq u$ and $v > u$. Let $v \leq u$ and $0 \leq t \leq t_* + \vartheta_1$. Then the functions λ_0, λ_v and λ_u are continuously differentiable and by Taylor series we obtain the estimates

$$\int_0^{t_* + \vartheta_1} |A_n(v, t)| dt \leq \frac{Cv^2}{n^2}, \quad \int_0^{t_* + \vartheta_1} |B_n(v, u, t)| dt \leq \frac{C(v^2 + vu)}{n^2}. \quad (11)$$

The similar estimates we have on the interval $[t_* + \vartheta_1 + u\varphi_n, \tau]$. We have as well the estimate

$$\int_{t_* + \vartheta_1 + v\varphi_n}^{t_* + \vartheta_1 + u\varphi_n} |A_n(v, t)| dt \leq \frac{Cv^2(u-v)}{n^3}, \quad (12)$$

$$\int_{t_* + \vartheta_1 + v\varphi_n}^{t_* + \vartheta_1 + u\varphi_n} |B_n(v, u, t)| dt \leq \frac{Cv^2(u-v)}{n^3} + \frac{Cv(u-v)}{n^2}. \quad (13)$$

The main contribution is done by the integrals

$$\begin{aligned} & n \int_{t_* + \vartheta_1}^{t_* + \vartheta_1 + v\varphi_n} \left[\exp\left(i\mu \ln \frac{\lambda_v}{\lambda_0}\right) - 1 \right] \lambda_u dt - i\mu n \int_{t_* + \vartheta_1}^{t_* + \vartheta_1 + v\varphi_n} [\lambda_v - \lambda_0] dt \\ &= v \left[\exp\left(i\mu \ln \frac{\lambda_-}{\lambda_+}\right) - 1 \right] \frac{\lambda_-}{\lambda_+} - iv\mu \left[\frac{\lambda_-}{\lambda_+} - 1 \right] + o(1) \\ &\longrightarrow v [\exp(i\mu \ln \rho) - 1] \rho - iv\mu [\rho - 1]. \end{aligned}$$

Therefore we obtain, for $v \leq u$,

$$\begin{aligned} \mathbf{E}_{\vartheta_1 + u\varphi_n} \exp\{i\mu \ln Z_n(v)\} &\longrightarrow \exp\left\{v \left(\exp(i\mu \ln \rho) - 1\right) \rho - i\mu(\rho - 1)\right\} \\ &= \mathbf{E}_u \exp\{i\mu \ln Z(v, u)\}. \end{aligned}$$

Now we consider the case when $v \geq u$. Similarly as before, we obtain the convergence ,

$$n \left(\int_0^{t_* + \vartheta_1} + \int_{t_* + \vartheta_1 + v\varphi_n}^{\tau} \right) (|A_n(v, t)| + |B_n(v, u, t)|) dt \longrightarrow 0.$$

For the intervals $(t_* + \vartheta_1, t_* + \vartheta_1 + u\varphi_n)$ and $(t_* + \vartheta_1 + u\varphi_n, t_* + \vartheta_1 + v\varphi_n)$, we can write

$$\begin{aligned} & n \int_{t_* + \vartheta_1}^{t_* + \vartheta_1 + u\varphi_n} [A_n(v, t) - i\mu B_n(v, u, t)] dt \\ &\longrightarrow \frac{u}{\lambda_+} \left(\exp\left(i\mu \ln \frac{\lambda_-}{\lambda_+}\right) - 1 \right) \lambda_- - i\mu(\lambda_- - \lambda_+) \\ &= u [(\exp\{i\mu \ln \rho\} - 1) \rho - i\mu(\rho - 1)] \end{aligned}$$

and

$$\begin{aligned} & n \int_{t_* + \vartheta_1 + u\varphi_n}^{t_* + \vartheta_1 + v\varphi_n} [A_n(v, t) - i\mu B_n(v, u, t)] dt \\ &\longrightarrow \frac{v-u}{\lambda_+} \left(\exp\left(i\mu \ln \frac{\lambda_-}{\lambda_+}\right) - 1 \right) \lambda_+ - i\mu(\lambda_- - \lambda_+). \end{aligned}$$

So we get for $v > u$,

$$\begin{aligned}
& \mathbf{E}_{\vartheta_1+u\varphi_n} \exp \{i\mu \ln Z_n(v)\} \\
& \longrightarrow \exp \left(u \left(\exp \{i\mu \ln \rho\} - 1 \right) \rho - iu\mu(\rho - 1) \right. \\
& \quad \left. + (v - u) \left((\exp \{i\mu \ln \rho\} - 1) - i\mu(\rho - 1) \right) \right) \\
& = \mathbf{E}_u \exp \{i\mu \ln Z(v, u)\}.
\end{aligned}$$

Therefore the one-dimensional distributions of the random process $Z_n(\cdot)$ converge to those of $Z(\cdot, u)$. Similarly we can prove the convergence of the finite-dimensional distributions of $Z_n(\cdot)$ to the those of $Z(\cdot, u)$. For example, in the case of two-dimensional distributions we can show, that $(v_1 < v_2 < u)$,

$$\begin{aligned}
& \mathbf{E}_{\vartheta_1+u\varphi_n} \exp \{i\mu_1 \ln Z_n(v_1) + i\mu_2 \ln Z_n(v_2)\} \\
& \rightarrow \exp \left\{ (v_2 - v_1) \left[\rho (\exp \{i\mu_2 \ln \rho\} - 1) - i\mu_2(\rho - 1) \right] \right. \\
& \quad \left. + v_1 \left[\rho (\exp \{i(\mu_1 + \mu_2) \ln \rho\} - 1) - i(\mu_1 + \mu_2)(\rho - 1) \right] \right\} \\
& = \mathbf{E}_u \exp \{i\mu_1 \ln Z(v_1, u) + i\mu_2 \ln Z(v_2, u)\}.
\end{aligned}$$

Similarly can be proved the convergence of the finite-dimensional distributions.

Further, we can write (under the alternative),

$$Z_n(v) = Z_n(u) \tilde{Z}_n(v),$$

where

$$\tilde{Z}_n(v) = \frac{d\mathbf{P}_{\vartheta_1+v\varphi_n}}{d\mathbf{P}_{\vartheta_1+u\varphi_n}}, \quad Z_n(u) = \frac{d\mathbf{P}_{\vartheta_1+u\varphi_n}}{d\mathbf{P}_{\vartheta_1}}.$$

Note that $Z_n(u)$ does not depend of v and we have the convergence

$$Z_n(u) \Longrightarrow Z(u) = \exp \{ \ln \rho x_*(u) - u(\rho - 1) \},$$

where $x_*(u)$ is Poisson process with the intensity function ρ . Therefore to prove (10) it is sufficient to study the convergence of the measures induced by the random process $\tilde{Z}_n(v), v \geq 0$.

Lemma 2. *Let conditions **S** be fulfilled. Then there exists a constant $C > 0$, such that*

$$\mathbf{E}_{\vartheta_1+u\varphi_n} |\tilde{Z}_n^{1/2}(v_1) - \tilde{Z}_n^{1/2}(v_2)|^2 \leq C |v_1 - v_2| \quad (14)$$

for all $v_1, v_2 \in U_n^+$.

Proof. According to [9, Lemma 1.1.5], we have, for $v_1 > v_2 > 0$,

$$\begin{aligned}
& \mathbf{E}_{\vartheta_1+u\varphi_n} |\tilde{Z}_n^{1/2}(v_1) - \tilde{Z}_n^{1/2}(v_2)|^2 \\
& \leq \int_0^{n\tau} \left(\frac{\lambda^{1/2}(t - \vartheta_1 - v_1\varphi_n)}{\lambda^{1/2}(t - \vartheta_1 - u\varphi_n)} - \frac{\lambda^{1/2}(t - \vartheta_1 - v_2\varphi_n)}{\lambda^{1/2}(t - \vartheta_1 - u\varphi_n)} \right)^2 \lambda(t - \vartheta_1 - u\varphi_n) dt \\
& = n \int_0^\tau (\lambda^{1/2}(t - \vartheta_1 - v_1\varphi_n) - \lambda^{1/2}(t - \vartheta_1 - v_2\varphi_n))^2 dt \\
& = n \left(\int_0^{t_*+v_2\varphi_n} + \int_{t_*+v_2\varphi_n}^{t_*+v_1\varphi_n} + \int_{t_*+v_1\varphi_n}^\tau \right) (\lambda_{v_1}^{1/2} - \lambda_{v_2}^{1/2})^2 dt \\
& = n(I_1 + I_2 + I_3).
\end{aligned}$$

As the functions λ_{v_1} and λ_{v_2} are continuously differentiable on the intervals $[0, t_* + v_2\varphi_n]$ and $[t_* + v_1\varphi_n, \tau]$, we can write

$$\lambda^{\frac{1}{2}}(\vartheta_1 + v_1\varphi_n, t) - \lambda^{\frac{1}{2}}(\vartheta_1 + v_2\varphi_n, t) = \frac{(v_1 - v_2)\varphi_n}{2} \frac{\dot{\lambda}(\tilde{\vartheta}_v, t)}{\lambda^{\frac{1}{2}}(\tilde{\vartheta}_v, t)}$$

Therefore

$$\begin{aligned}
n(I_1 + I_3) & \leq n\varphi_n^2 \frac{(v_1 - v_2)^2}{4} \left(\int_0^{t_*+v_2\varphi_n} + \int_{t_*+v_1\varphi_n}^\tau \right) \frac{\dot{\lambda}^2(\vartheta_v, t)}{\lambda(\vartheta_v, t)} dt \\
& \leq \frac{C}{n\lambda_+^2} |v_1 - v_2|^2 \leq C |v_1 - v_2|
\end{aligned}$$

because $|v_1 - v_2| \leq C\lambda_+ n$.

The function λ is bounded therefore we have the estimate

$$nI_2 \leq n \frac{|v_1 - v_2|}{n\lambda_+} C = \frac{C}{\lambda_+} |v_1 - v_2|$$

and the inequality (14) holds with some constant $C > 0$.

Lemma 3. *Let conditions \mathbf{S} be fulfilled. Then there exists a constant $k^* > 0$ such that*

$$\mathbf{E}_{\vartheta_1+u\varphi_n} \tilde{Z}_n^{1/2}(v) \leq \exp\{-k^* |v - u|\} \quad (15)$$

for all $v \in U_n^+$.

Proof. According to [9, Lemma 1.1.5], we have

$$\begin{aligned}
& \mathbf{E}_{\vartheta_1+u\varphi_n} \tilde{Z}_n^{1/2}(v) \\
&= \exp \left\{ -\frac{n}{2} \int_0^\tau \left(\frac{\lambda^{1/2}(t - \vartheta_1 - v\varphi_n)}{\lambda^{1/2}(t - \vartheta_1 - u\varphi_n)} - 1 \right)^2 \lambda(t - \vartheta_1 - u\varphi_n) dt \right\} \\
&= \exp \left\{ -\frac{n}{2} \int_0^\tau \left(\lambda^{1/2}(t - \vartheta_1 - v\varphi_n) - \lambda^{1/2}(t - \vartheta_1 - u\varphi_n) \right)^2 dt \right\} \\
&= \exp \left\{ -\frac{n}{2} F_n(u, v) \right\}
\end{aligned}$$

with obvious notation. Let us consider two cases $D = \{v : |v - u| < \delta n \lambda_+\}$ and $D^c = \{v : |v - u| \geq \delta n \lambda_+\}$ separately. Here δ is some positive constant which we choose later. For simplicity we suppose that $v > u$.

If $v \in D$ then

$$\begin{aligned}
nF_n(u, v) &\geq n \int_{t_*+u\varphi_n}^{t_*+\vartheta_1+v\varphi_n} \left[\lambda^{1/2}(t - \vartheta_1 - v\varphi_n) - \lambda^{1/2}(t - \vartheta_1 - u\varphi_n) \right]^2 dt \\
&\geq \frac{|v - u|}{\lambda_+} \inf_{t_*+u\varphi_n \leq s \leq t_*+v\varphi_n} \left[\lambda^{1/2}(s - v\varphi_n) - \lambda^{1/2}(s - u\varphi_n) \right]^2 \\
&\geq \frac{|v - u|}{2\lambda_+} \left(\sqrt{\lambda_-} - \sqrt{\lambda_+} \right)^2 = \frac{|v - u|}{2} (\sqrt{\rho} - 1)^2
\end{aligned}$$

for sufficiently small δ .

Further, note that for any $\nu > 0$

$$g(\nu) = \inf_{|s-s_0|>\nu} \int_0^\tau \left[\sqrt{\lambda(t - \vartheta_1 - s)} - \sqrt{\lambda(t - \vartheta_1 - s_0)} \right]^2 dt > 0$$

because if $g(\nu)=0$, then for some s_* we have $\lambda(t - \vartheta_1 - s_*) = \lambda(t - \vartheta_1 - s_0)$ for all $t \in [0, \tau]$, but this equality for discontinuous $\lambda(\cdot)$ and all t is impossible. Hence for the values $v \in D^c$ we have

$$nF_n(v, u) \geq ng(\delta) \geq \frac{g(\delta)|v - u|}{C}$$

because $|v - u| \leq Cn$.

Therefore (15) is proved.

The presented estimates (14), (15) and Lemma 1 allow us to finish the proof following the same lines as it was done in [8], Section 4.4.3.

4.2 GLRT

The GLRT is based on the statistic

$$Q_n(X^n) = \sup_{\vartheta \geq \vartheta_1} L(\vartheta, \vartheta_1, X^n) = \max \left[L(\hat{\vartheta}_n +, \vartheta_1, X^n), L(\hat{\vartheta}_n -, \vartheta_1, X^n) \right]$$

and is of the form

$$\psi_n(X^n) = \mathbb{I}_{\{Q_n(X^n) > h_\varepsilon\}}.$$

The threshold h_ε we define with the help of the convergence (under \mathcal{H}_1)

$$Q_n(X^n) = \sup_{v \in \mathbb{U}_n^+} Z_n(v) \implies \sup_{v > 0} Z(v) = \hat{Z}.$$

Hence $h_\varepsilon = h_\varepsilon(\rho)$ is solution of the equation

$$\mathbf{P} \left\{ \hat{Z} > h_\varepsilon \right\} = \varepsilon.$$

Let us fix an alternative $u > 0$, then for the power function we have

$$\begin{aligned} \beta(\psi_n, u) &= \mathbf{E}_{\vartheta_1 + u\varphi_n} \psi_n(X^n) = \mathbf{P}_{\vartheta_1 + u\varphi_n} \left\{ \sup_{v > 0} Z_n(v) > h_\varepsilon \right\} \\ &\rightarrow \mathbf{P}_u \left\{ \sup_{v > 0} Z(v, u) > h_\varepsilon \right\}, \end{aligned}$$

where

$$Z(v, u) = \exp \{ \ln \rho x_*(v, u) - (\rho - 1)v \}, \quad v \geq 0$$

with the Poisson process $x_*(v, u), v \geq 0$ of the intensity function

$$\mu(u, v) = \rho \mathbb{I}_{\{v < u\}} + \mathbb{I}_{\{v \geq u\}}, \quad v \geq 0.$$

Let us put $Y(v) = \ln \rho x_*(v, u) - (\rho - 1)v$, then we can write

$$\begin{aligned} &\sup_{v > 0} [\ln \rho x_*(v, u) - (\rho - 1)v] \\ &= \max \left(\sup_{0 < v < u} Y(v), Y(u) + \sup_{v \geq u} [Y(v) - Y(u)] \right). \end{aligned}$$

Note that the Poisson process $\tilde{x}(v - u) = x_*(v) - x_*(u), v \geq u$ is independent of $x_*(u)$ and $x_*(v, u), 0 \leq v \leq u$. Hence we can write the representation of the limit power as follows

$$\beta(\hat{\psi}, u) = \mathbf{P}_u \left\{ \max \left(\sup_{0 < v < u} Z(v), Z_*(u) \tilde{Z} \right) > h_\varepsilon \right\}. \quad (16)$$

The random variable $\tilde{Z} = \sup_{v \geq 0} \exp \{ \ln \rho \tilde{x}_*(v) - (\rho - 1)v \}$ is independent of $Z(v), 0 \leq v \leq u$ and $Z(u)$. Therefore this expression can be used for the numerical simulation of the power function. It simplifies the simulations because the simulated values of \tilde{Z} can be used many times for the different values of u .

4.3 Wald test

The Wald test is based on the MLE $\hat{\vartheta}_n$. We already know that

$$\varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_1) \Rightarrow \hat{v},$$

where \hat{v} is solution of the equation

$$\max[Z_*(\hat{v}+), Z_*(\hat{v}-)] = \sup_{v>0} Z_*(v).$$

The Wald test is

$$\psi_n(X^n) = \mathbb{I}_{\{\varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_1) > g_\varepsilon\}}.$$

The threshold $g_\varepsilon = g_\varepsilon(\rho)$ is solution of the equation

$$\mathbf{P}\{\hat{v} > g_\varepsilon\} = \varepsilon.$$

For the power function we have (below $\vartheta_u = \vartheta_1 + u\varphi_n$)

$$\begin{aligned} \beta(\psi_n, u) &= \mathbf{E}_{\vartheta_u} \psi_n(X^n) = \mathbf{P}_{\vartheta_u} \left\{ \varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_1) > h_\varepsilon \right\} \\ &= \mathbf{P}_{\vartheta_u} \left\{ \sup_{\varphi_n^{-1}(\theta - \vartheta_1) > h_\varepsilon} L(\theta, X^n) > \sup_{\varphi_n^{-1}(\theta - \vartheta_1) \leq h_\varepsilon} L(\theta, X^n) \right\} \\ &= \mathbf{P}_{\vartheta_u} \left\{ \sup_{\varphi_n^{-1}(\theta - \vartheta_1) > h_\varepsilon} \frac{L(\theta, X^n)}{L(\vartheta_1, X^n)} > \sup_{\varphi_n^{-1}(\theta - \vartheta_1) \leq h_\varepsilon} \frac{L(\theta, X^n)}{L(\vartheta_1, X^n)} \right\} \\ &= \mathbf{P}_{\vartheta_u} \left\{ \sup_{v > h_\varepsilon} Z_n(v) > \sup_{v \leq h_\varepsilon} Z_n(v) \right\} \\ &\longrightarrow \mathbf{P}_{\vartheta_u} \left\{ \sup_{v > h_\varepsilon} Z(v, u) > \sup_{v \leq h_\varepsilon} Z(v, u) \right\} = \mathbf{P}_{\vartheta_u} \{\hat{v}_u > h_\varepsilon\}, \end{aligned}$$

where the random variable \hat{v}_u is defined by the equation

$$\sup_{v \geq 0} Z(v, u) = \max(Z(\hat{v}_u +, u), Z(\hat{v}_u -, u))$$

We can write another representation as well

$$\begin{aligned} \beta(\psi_n, u) &= \mathbf{E}_{\vartheta_u} \psi_n(X^n) = \mathbf{P}_{\vartheta_u} \left\{ \varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_u) + u > h_\varepsilon \right\} \\ &\longrightarrow \mathbf{P}\{\hat{v}_* > h_\varepsilon - u\} \end{aligned}$$

where \hat{v}_* is the solution of the equation

$$\max[Z^*(\hat{v}_* +), Z^*(\hat{v}_* -)] = \sup_{v > -u} Z^*(v),$$

where the random process $Z^*(v)$ is defined on the interval $[-u, \infty)$ and the corresponding Poisson process $x_*(v)$, $v \geq -u$ has the intensity function $\mu(v) = \rho \mathbb{I}_{\{-u \leq v < 0\}} + \mathbb{I}_{\{v \geq 0\}}$.

4.4 Bayesian tests

Suppose that the parameter ϑ is a random variable with known probability density $p(\theta)$, $\vartheta_1 \leq \theta < b$. This function is supposed to be continuous and positive. We consider two tests.

The first one is like Wald test but based on the BE $\tilde{\vartheta}_n$

$$\tilde{\psi}_n(X^n) = \mathbb{I}_{\{\varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1) > k_\varepsilon\}}, \quad \tilde{\vartheta}_n = \frac{\int_{\vartheta_1}^b \theta p(\theta) L(\theta, \vartheta_1, X^n) d\theta}{\int_{\vartheta_1}^b \theta L(\theta, \vartheta_1, X^n) d\theta}.$$

The properties of the likelihood ratio established in Lemmae 1-3 allow us to justify the limit

$$\mathbf{E}_{\vartheta_1} \mathbb{I}_{\{\varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1) > k_\varepsilon\}} \longrightarrow \mathbf{P}_{\vartheta_1} \{\tilde{v} > k_\varepsilon\}, \quad \tilde{v} = \frac{\int_0^\infty v Z(v) dv}{\int_0^\infty Z(v) dv},$$

where the Poisson process $x_*(v)$, $v \geq 0$ in $Z(v)$, as before, has the unit intensity function. The proof follows from the general results concerning the bayesian estimators described in [7] (see as well [8]).

For the power function the limit is obtained from the following convergence :

$$\varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1) = \varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_u) + u \implies \frac{\int_{-u}^\infty v Z^*(v) dv}{\int_{-u}^\infty Z^*(v) dv} + u.$$

Recall that the Poisson process $x_*(v)$, $v \geq 0$ in the definition of $Z^*(v)$ has the intensity function $\rho \mathbb{I}_{\{-u \leq v < 0\}} + \mathbb{I}_{\{v \geq 0\}}$.

We have the convergence of $Z_n(v)$, $v \geq 0$ to $Z(v, u)$, $v \geq 0$ (under alternatives) hence

$$\beta(\tilde{\psi}_n, u) = \mathbf{P}_{\vartheta_u} \left\{ \varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1) > k_\varepsilon \right\} = \mathbf{P}_{\vartheta_u} \left\{ \frac{\int_0^\infty v Z(v, u) dv}{\int_0^\infty Z(v, u) dv} > k_\varepsilon \right\},$$

where the Poisson process $x_*(v)$, $v \geq 0$ in $Z(v, u)$ has the intensity (8).

The thresholds and power function are obtained by the numerical simulations.

The second bayesian test is the test which minimizes the mean error. We have

$$\varphi_n^{-1} \tilde{L}(X^n) = \varphi_n^{-1} \int_{\vartheta_1}^b p(\theta) \frac{L(\theta, X^n)}{L(\vartheta_1, X^n)} d\theta \implies p(\vartheta_1) \int_0^\infty Z(v) dv.$$

Hence the test

$$\tilde{\psi}_n(X^n) = \mathbb{I}_{\{R_n > m_\varepsilon\}}, \quad R_n = \frac{\tilde{L}(X^n)}{\varphi_n p(\vartheta_1)}$$

with the threshold m_ε satisfying the equation

$$\mathbf{P} \left\{ \int_0^\infty \exp \{ \ln \rho x_*(v) - (\rho - 1)v \} dv > m_\varepsilon \right\} = \varepsilon$$

is bayesian and belongs to the class \mathcal{K}_ε .

4.5 Simulations

We consider n independent observations $X_j^{(n)} = \{X_j^{(n)}(t), t \in [0, 4]\}$; $j = 1, \dots, n$ of a Poisson process of intensity function

$$\lambda(t, \vartheta) = \lambda(t - \vartheta) = 3 - 2 \cos^2(t - \vartheta) \mathbb{I}_{\{t \geq \vartheta\}}, \quad 0 \leq t \leq 4.$$

We take $\vartheta_1 = 3$ and $b = 4$, i.e., $\vartheta \in [3, 4]$. Therefore we obtain the values $\lambda_+ = 1$, $\lambda_- = 3$ and $\rho = \frac{\lambda_-}{\lambda_+} = 3$. The log-likelihood ratio is

$$\begin{aligned} \ln Z_n(v) &= \sum_{j=1}^n \int_3^{3+v/n} \ln \frac{3}{3 - 2 \cos^2(t - 3)} dX_j(t) \\ &\quad + \sum_{j=1}^n \int_{3+v/n}^4 \ln \frac{3 - 2 \cos^2(t - 3 - v/n)}{3 - 2 \cos^2(t - 3)} dX_j(t) \\ &\quad - v - \frac{n}{2} \sin(2) + \frac{n}{2} \sin(2(1 - v/n)). \end{aligned}$$

The realization of the likelihood ratio $Z_n(v)$ and of its zoom are given on the Fig. 5.

Here Fig. 5

We recall that in this case the limit of the likelihood ratio is

$$Z(v) = \exp \{ \ln 3 x_*(v) - 2v \},$$

where $x_*(v), v \geq 0$ is the Poisson process of unit intensity.

Using the limit expression $Z(\cdot)$, we obtain the threshold g_ε of the GLRT as solution of the equation

$$\mathbf{P} \left\{ \hat{Z} > g_\varepsilon \right\} = \varepsilon.$$

It was convinient for simulations to transform the limit process as follows

$$\exp \left[\ln 3 \left(x_*(v) - \frac{2}{\ln 3} v \right) \right] = \exp \left\{ \ln 3 \left[\Pi \left(\frac{2}{\ln 3} v \right) - \frac{2}{\ln 3} v \right] \right\}$$

where $\Pi(\cdot)$ is the Poisson process of constant intensity $\gamma = \frac{\ln 3}{2} < 1$.

Hence we can choose the threshold of GLRT by following relation

$$\mathbf{P} \left\{ \sup_{v>0} Z(v) > g_\varepsilon \right\} = \mathbf{P} \left\{ \sup_{t>0} [\Pi(t) - t] > \frac{\ln g_\varepsilon}{\ln 3} \right\} = \varepsilon.$$

The distribution of $\sup_{t>0} [\Pi(t) - t]$ is given by the well-known formula obtained by Pyke [15]

$$\mathbf{P} \left\{ \sup_{t>0} [\Pi(t) - t] \geq x \right\} = \sum_{m>x} \frac{(m-x)^m}{m!} (\gamma e^{-\gamma})^m e^{\gamma x} (1-\gamma). \quad (17)$$

Note that we have as well the analitic expression for the distribution of the random variable $\hat{t} = \arg \sup_{t \geq 0} [\Pi(t) - t]$ obtained by Pflug [14]

$$\mathbf{P} \{ \hat{v}_* > h_\varepsilon \} = \mathbf{P} \left\{ \arg \sup_{t \geq 0} [\Pi(t) - t] > \frac{\ln 3}{2} h_\varepsilon \right\} = \varepsilon.$$

This distribution is

$$\mathbf{P} \{ \hat{t} < z \} = \mathbf{P} \left\{ \sum_{k=1}^{\nu} \eta_k < z \right\}$$

where ν is a geometric random variable, independent of $\eta_k, k \geq 0$

$$\mathbf{P} \{ \nu = i \} = (1-\gamma)\gamma^i; i = 0, 1, \dots$$

($\sum_{k=1}^0 Q_k$ is set to zero) and $\{Q_k\}; k = 1, 2, \dots$ is an i.i.d sequence with common distribution

$$F(x) = \mathbf{P} \{ Q_k \leq x \} = \frac{1}{\gamma} \left[1 - (1-\gamma)e^{-\gamma x} \sum_{j=0}^{[x]-1} \frac{(\gamma x)^j}{j!} - e^{-\gamma x} \frac{(\gamma x)^{[x]}}{[x]!} \right].$$

The numerical slution of the corresponfing equation for the threshold of Wald test is not easy and we decided to obtain this threshold by the Monte Carlo simulations.

The thresholds are presented on the following table.

ε	0.01	0.05	0.10	0.20	0.40	0.50
$\ln h_\varepsilon$	4.242	2.607	1.922	1.120	0.573	0.191
g_ε	5.990	3.556	2.078	1.045	0.329	0.099
k_ε	6.669	3.937	2.983	2.132	1.402	1.196

Table 2: Thresholds of GLRT, WT and BT1.

Here Fig. 6

It is interesting to compare the studied tests with the Neyman-Pearson test. Of course, it is impossible to construct the N-PT in our problem because the value u under alternative is unknown. Nevertheless its power function shows an upper bound and the distance between it and the power functions of studied tests provides an important information. Let us fix some $u_1 > 0$ and introduce the N-PT

$$\psi_n^*(X^n) = \mathbb{I}_{\{Z_n(u_1) > d_\varepsilon\}} + q_\varepsilon \mathbb{I}_{\{Z_n(u_1) = d_\varepsilon\}},$$

where $d_\varepsilon, q_\varepsilon$ are solutions of the equation

$$\mathbf{P}_{\vartheta_1}(Z_*(u_1) > d_\varepsilon) + q_\varepsilon \mathbf{P}_{\vartheta_1}(Z_*(u_1) = d_\varepsilon) = \varepsilon.$$

Denote $D_\varepsilon = (\ln d_\varepsilon + (\rho - 1)u_1) / \ln \rho$ and rewrite this equation as

$$\mathbf{P}_{\vartheta_1}(x_*(u_1) > D_\varepsilon) + q_\varepsilon \mathbf{P}_{\vartheta_1}(x_*(u_1) = D_\varepsilon) = \varepsilon.$$

We have

$$\mathbf{P}_{\vartheta_1}(x_*(u_1) = D_\varepsilon) = \mathbf{P}_{\vartheta_1}(x_*(u_1) > D_\varepsilon -) - \mathbf{P}_{\vartheta_1}(x_*(u_1) > D_\varepsilon),$$

where the Poisson random variable $x_*(\cdot)$ has the parameter u_1 . The quantities D_ε and q_ε can be calculated.

Similar calculation yields the limit power function

$$\beta(\psi_n^*, u_1) \rightarrow \mathbf{P}_{u_1}(x_*(u_1) > D_\varepsilon) + q_\varepsilon \mathbf{P}_{u_1}(x_*(u_1) = D_\varepsilon).$$

where the Poisson random variable $x_*(u_1)$ has the parameter ρu_1 . The results of simulations are presented on the Fig.7.

Here Fig. 7

We considered two cases with the values $\varepsilon = 0.05$ and 0.4 . In both cases the GLRT shows the better power functions.

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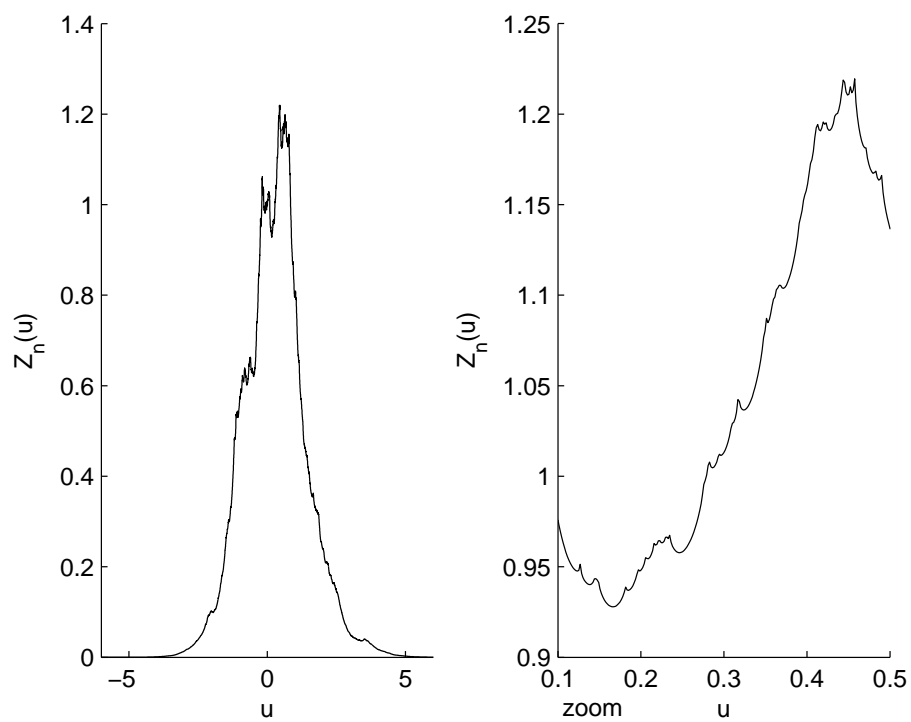


Figure 1: Realizations of $Z_n(u)$ with $\lambda(\vartheta, t) = 2 - |t - 1.5|^{0.4}$ and $n = 10^4$.

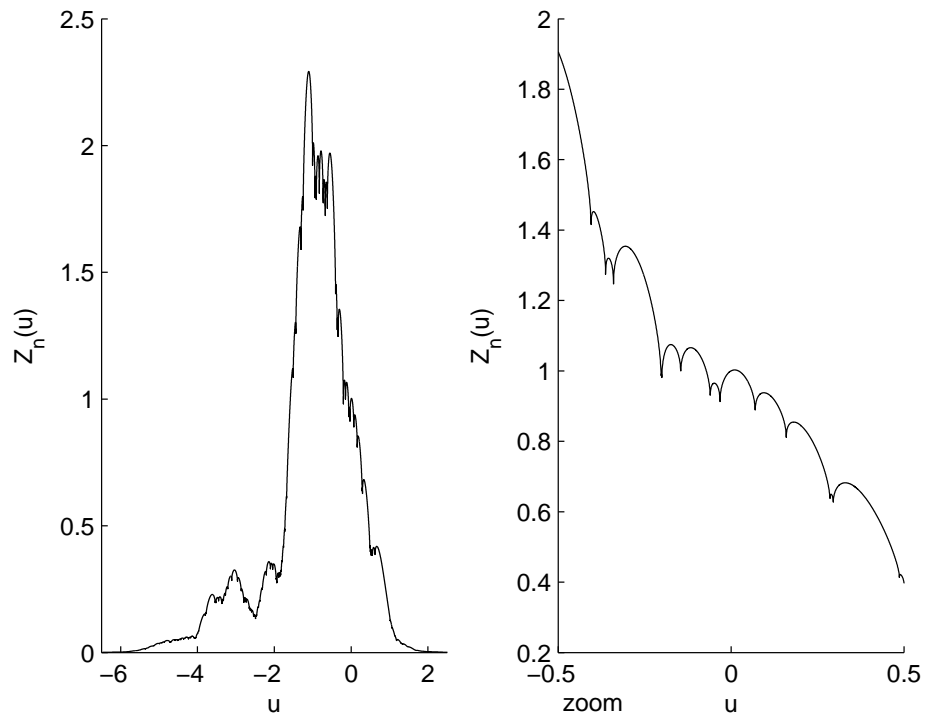


Figure 2: Some realizations of $Z_n(u)$ with $\lambda(\vartheta, t) = 0.5 + |t - 1.5|^{0.4}$ and $n = 1000$.

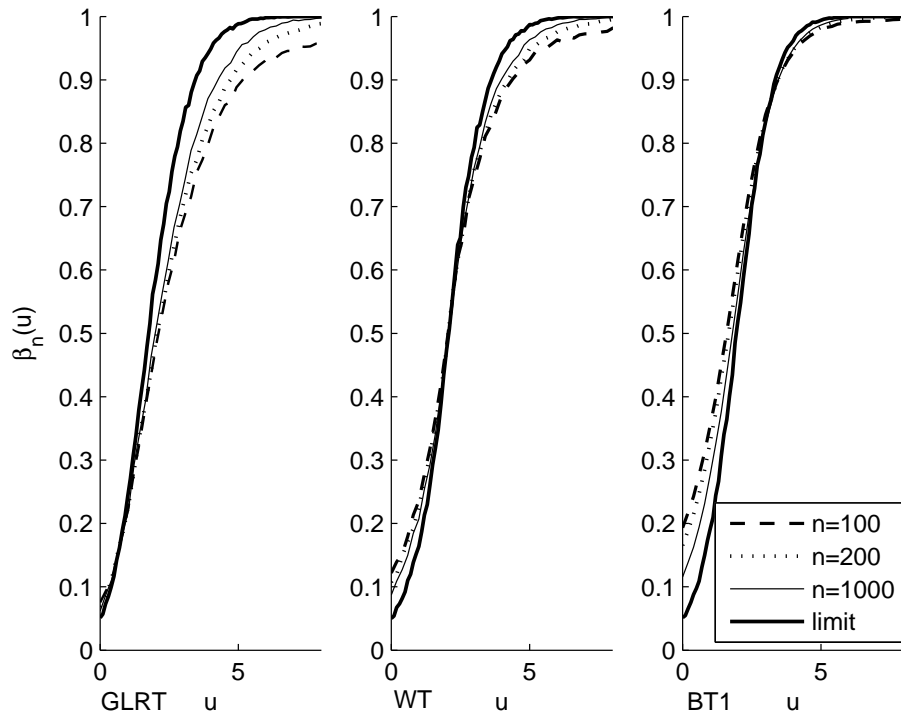


Figure 3: Power functions of GLRT, WT and BT1 in cusp case with $\lambda(\vartheta, t) = 2 - |t - \vartheta|^{0.4}$.

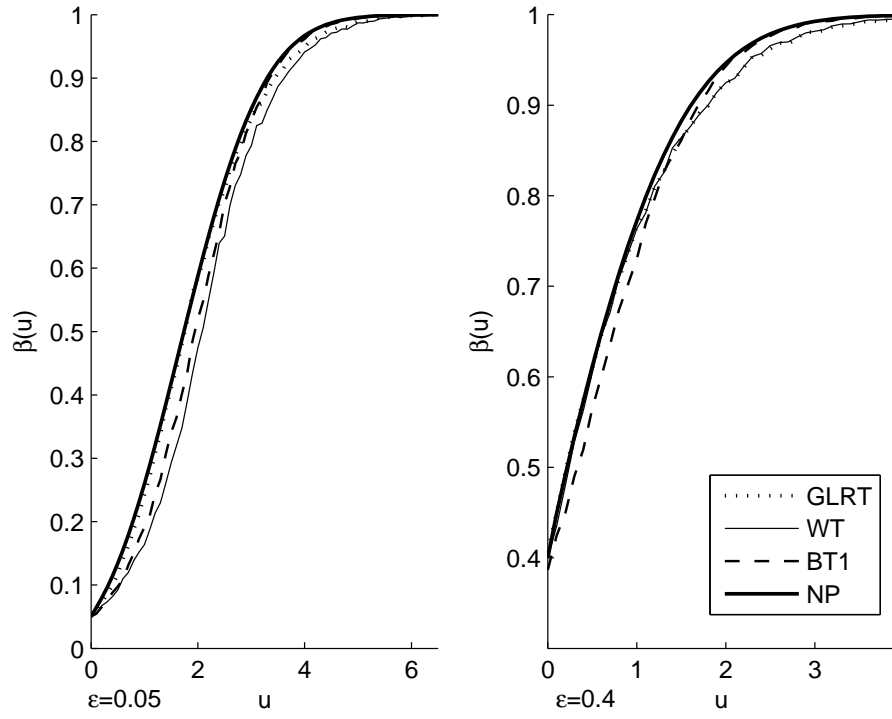


Figure 4: Comparison of limit power functions for cusp type with $\lambda(\vartheta, t) = 2 - |t - \vartheta|^{0.4}$.

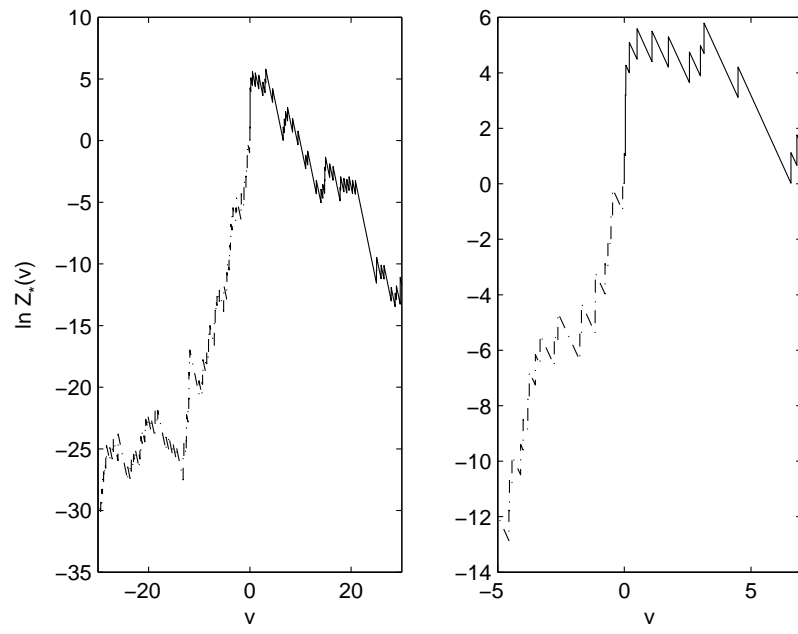


Figure 5: Realization of $\ln Z_*$

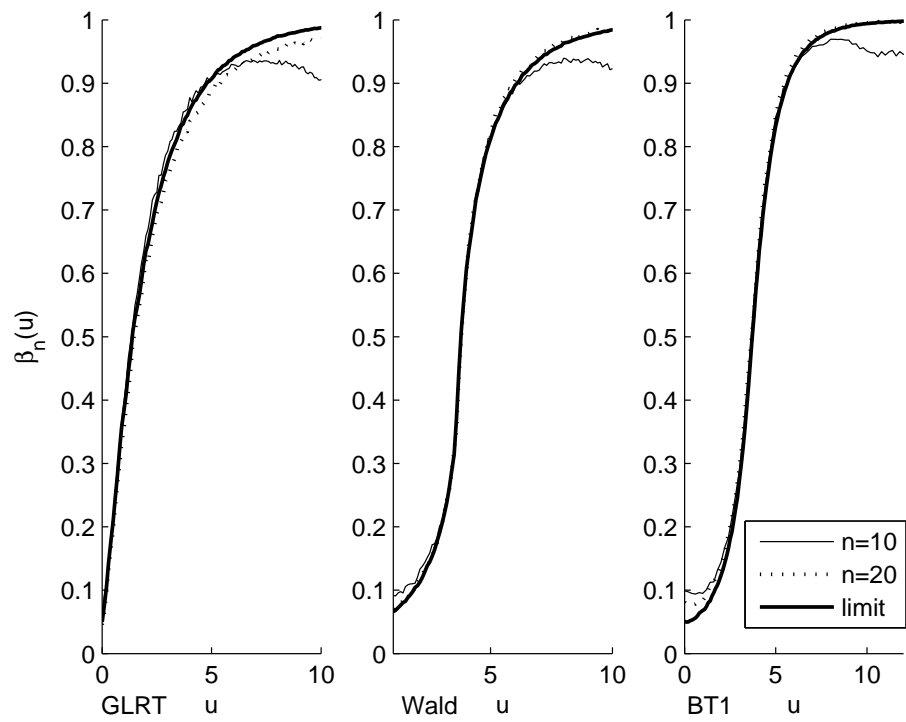


Figure 6: Power functions of GLRT, WT and BT1 in discontinuous case

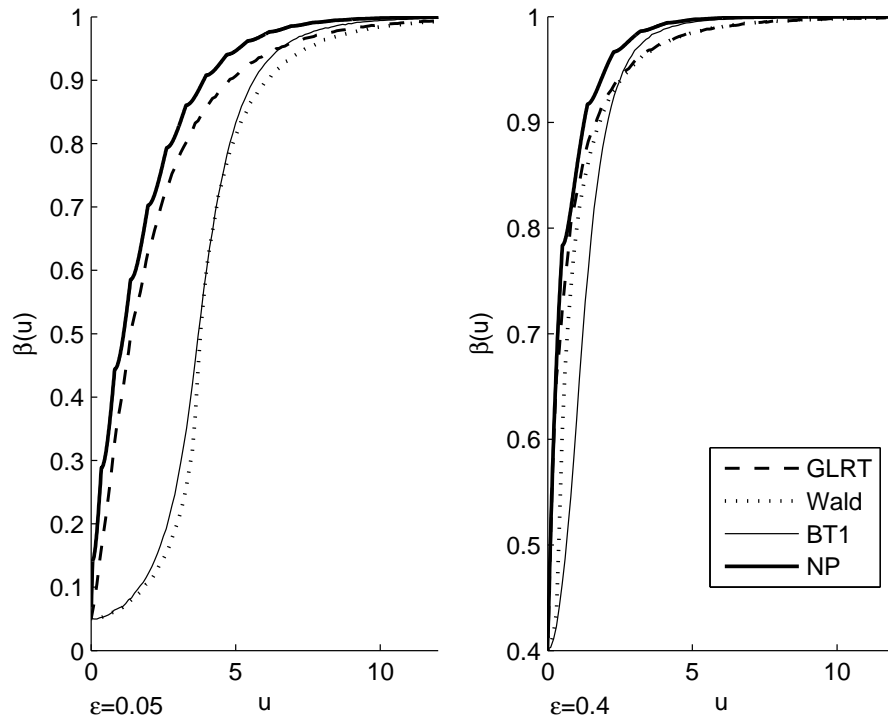


Figure 7: Comparison of different Power functions with $\rho = 3$.